Stable sets in {ISK4,wheel}-free graphs

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Algorithmic Graph Theory on the Adriatic Coast
Koper, Slovenia

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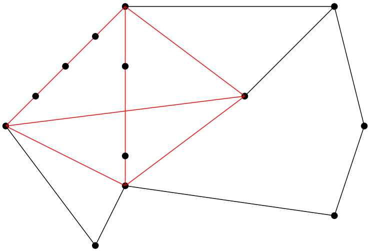
Definition

An *ISK4* in a graph G is an induced subdivision of K_4 in G. A graph is *ISK4-free* if it contains no induced subdivision of K_4.



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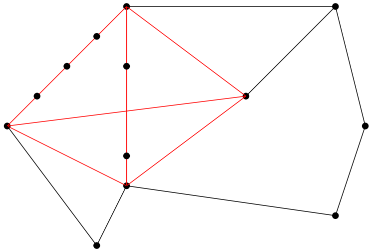
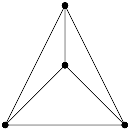
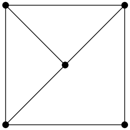
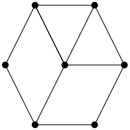
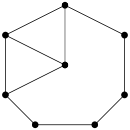


Remark: An ISK4-free graph is in particular K_4-free, so it has no cliques of size greater than 3.

Definition

A *wheel* is a graph that consists of a chordless cycle and an additional vertex that has at least three neighbors in the cycle. A graph is *wheel-free* if it contains no wheel as an induced subgraph.

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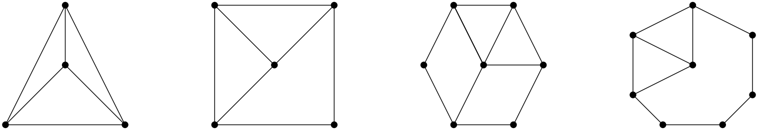

Definition

A graph is $\{ISK4, wheel\}$*-free* if it is both ISK4-free and wheel-free.

Theorem [Milanič, P., Trotignon, 2015+]

There is an algorithm with the following specifications:

- Input: A weighted {ISK4,wheel}-free graph (G, w) [a];
- Output: $\alpha(G, w)$ [b];
- Running time: $O(n^7)$, where $n = |V(G)|$.

[a] The *weight function* w assigns a non-negative integer weight $w(v)$ to each vertex v of G.

[b] $\alpha(G, w)$ is the maximum weight of a stable set (i.e. a set of pairwise non-adjacent vertices) of G with respect to w.

State of the art for **wheel-free** graphs:

1. recognition is NP-complete (Diot, Tavenas, Trotignon, 2014);
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State of the art for {**ISK4,wheel**}**-free** graphs:

1. decomposition theorem for {ISK4,wheel}-free graphs (Lévêque, Maffray, Trotignon, 2012);
2. polynomial-time recognition algorithm for {ISK4,wheel}-free graphs (Lévêque, Maffray, Trotignon, 2012);
3. {ISK4,wheel}-free graphs are 3-colorable + polynomial-time algorithm to 3-color them (Lévêque, Maffray, Trotignon, 2012).

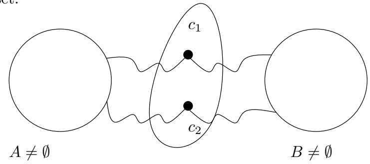
Theorem [Lévêque, Maffray, Trotignon, 2012]

If G is an {ISK4,wheel}-free graph, then either:

- G is a series-parallel graph[a], or
- G is the line graph of a chordless graph[b] of maximum degree at most three, or
- G is a complete bipartite graph, or
- G admits a clique-cutset, or
- G admits a proper 2-cutset.

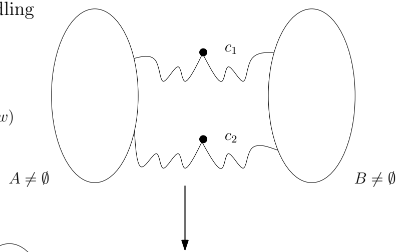
[a] series-parallel = no subdivision K_4 as a subgraph
[b] chordless = all cycles are induced

Proper 2-cutset:

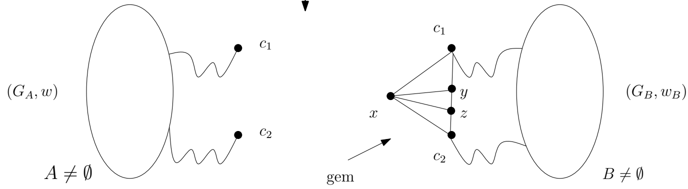


Neither $A \cup \{c_1, c_2\}$ nor $B \cup \{c_1, c_2\}$ induces a path between c_1 and c_2.

Attempt at handling proper 2-cutsets:



Need:
$\alpha(G_B, w_B) = \alpha(G, w)$



The weight of the gem gem vertices that lie inside a maximum weighted stable set of (G_B, w_B) is supposed to be equal to the weight of the part of a maximum weighted stable set of (G, w) that lies in (G_A, w).

$w_B(c_1) + w_B(c_2) = \alpha(G_A, w)$

$w_B(c_1) + w_B(z) = \alpha(G_A \setminus \{c_2\}, w)$

$w_B(c_2) + w_B(y) = \alpha(G_A \setminus \{c_1\}, w)$

$w_B(x) = \alpha(G_A \setminus \{c_1, c_2\}, w)$

$w_B(c_2) = w(c_2)$

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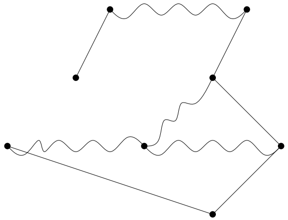
- We defined weighted trigraphs (we put weights on vertices and semi-adjacent pairs; motivated by proper 2-cutsets), and we constructed a polynomial-time algorithm that finds the maximum weight of a stable set in weighted {ISK4,wheel}-free trigraphs.
    - Since every weighted {ISK4,wheel}-free graph is a weighted {ISK4,wheel}-free trigraph, this will yield a polynomial-time algorithm that finds the maximum weight of a stable set in a weighted {ISK4,wheel}-free graph.

Definition

A *trigraph* is a generalization of a graph in which there are three types of adjacency:

- strongly-adjacent pairs ("edges"),
- strongly anti-adjacent pairs ("non-edges"),
- semi-adjacent pairs ("optional edges" or "pairs of undetermined adjacency").

An *adjacent pair* is a pair or strongly-adjacent or semi-adjacent vertices. An *anti-adjacent pair* is a pair of strongly anti-adjacent or semi-adjacent vertices.



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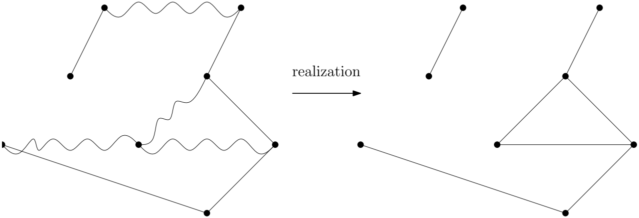
A *realization* of trigraph is any graph obtained by turning each semi-adjacent pair into an edge or a non-edge. So a trigraph with m semi-adjacent pairs has 2^m realizations.



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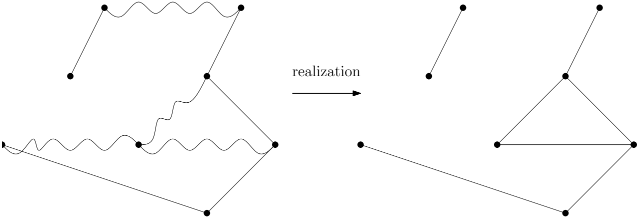
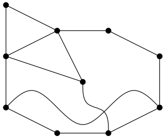


Definition

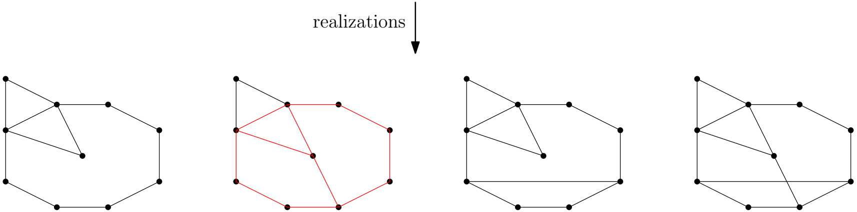
The *full realization* of trigraph is the graph obtained by turning all its semi-adjacent pairs into edges.

Definition

A trigraph is *ISK4-free* (resp. *wheel-free*, {*ISK4,wheel*}*-free*) if all its realizations are ISK4-free (resp. wheel-free, {ISK4,wheel}-free).

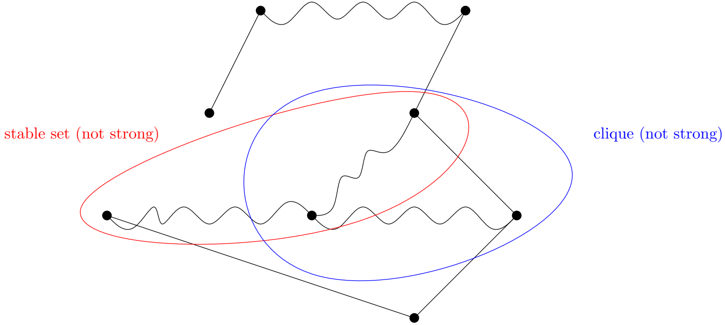


The trigraph is **not** wheel-free because it has a realization that is not wheel-free.



Definition

A *clique* in a trigraph is a set of pairwise adjacent (possibly semi-adjacent) vertices, and a *stable set* is a set of pairwise anti-adjacent (possibly semi-adjacent) vertices. A *strong clique* (resp. *strongly stable set*) is a clique (resp. stable set) with no semi-adjacent pairs.



We proved an "extreme decomposition theorem" that states that every $\{$ISK4,wheel$\}$-free trigraph is either "basic" or admits a "cutset" so that one of the "blocks of decomposition" is "basic."

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Definition

A trigraph is *basic* if it is either

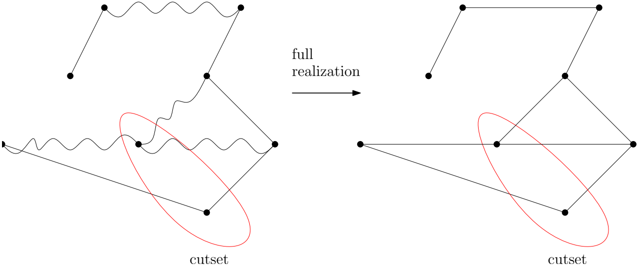
- a series-parallel trigraph (i.e. its full realization is a series-parallel graph), or
- a line trigraph[a] of a chordless graph of maximum degree at most three, or
- a complete bipartite graph.

---

[a] G is a *line trigraph* of a graph H if the full realization of G is the line graph of H, and no semi-adjacent pair of G is in a triangle.

Definition

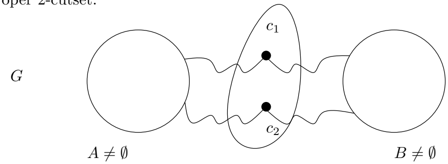
A trigraph is *connected* if its full realization is connected, and otherwise, it is *disconnected*. A *cutset* of a trigraph is a (possibly empty) set of vertices whose deletion yields a disconnected trigraph.



Theorem [Milanič, P., Trotignon, 2015+]

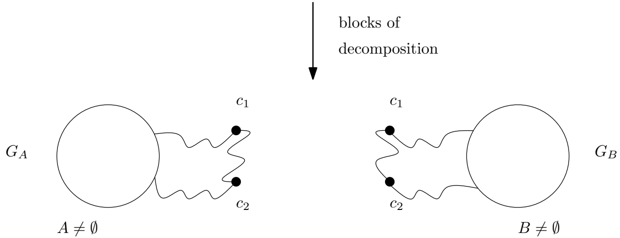
Every {ISK4,wheel}-free trigraph is either basic or admits a clique-cutset (i.e. a strong clique that is a cutset) or a proper 2-cutset s.t. one of the induced "blocks of decomposition" is basic.

Proper 2-cutset:



$c_1 c_2$ is an anti-adjacent pair (possibly semi-adjacent)

G is connected, and neither c_1 nor c_2 is a cut-vertex.



Fact: If G is {ISK4,wheel}-free, then so are G_A and G_B.

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Every {ISK4,wheel}-free trigraph is either basic or admits a clique-cutset (i.e. a strong clique that is a cutset) or a proper 2-cutset s.t. one of the induced “blocks of decomposition” is basic.

Proof: Imitate the proof of the decomposition theorem for ISK4-free graphs (Lévêque, Maffray, Trotignon, 2012). Generalize to trigraphs, but(!) consider only the wheel-free case.

- The “jump” to trigraphs doesn’t complicate the proof much; the restriction to the wheel-free case significantly simplifies it.

A bit of extra work to get the “extreme” decomposition theorem.

- This is algorithmic! There is a polynomial-time algorithm that, given an {ISK4,wheel}-free trigraph G, either determines that G is basic, or finds an “extreme decomposition” of G via a clique-cutset or a proper 2-cutset. Q.E.D.

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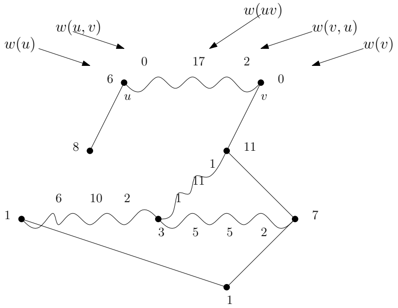
  - Thus, we can plug in a weighted {ISK4,wheel}-free graph (G, w) into our algorithm for trigraphs and get an "ordinary" $\alpha(G, w)$ for weighted graphs.

Definition

A *weighted trigraph* is an ordered pair (G, w) s.t. G is a trigraph, and w is a *weight function* for G s.t.

- to each vertex v of G, w assigns a non-negative integer weight $w(v)$, and
- to each semi-adjacent pair uv of G, w assigns three non-negative integer weights, $w(uv)$, $w(u, v)$, and $w(v, u)$, and these weights satisfy $w(u, v), w(v, u) \leq w(uv)$.

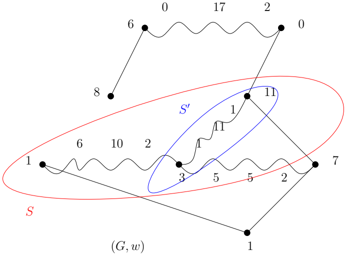


Definition

The *weight* of a set S of vertices in a weighted trigraph (G, w) is the sum of the following three quantities:

- the sum of all $w(u)$ s.t. $u \in S$;
- the sum of all $w(u, v)$ s.t. uv is a semi-adjacent pair of G with $u \in S$ and $v \notin S$;
- the sum of all $w(uv)$ s.t. uv is a semi-adjacent pair of G with $u, v \notin S$.

$\alpha(G, w)$ is the maximum weight of a stable set of (G, w).



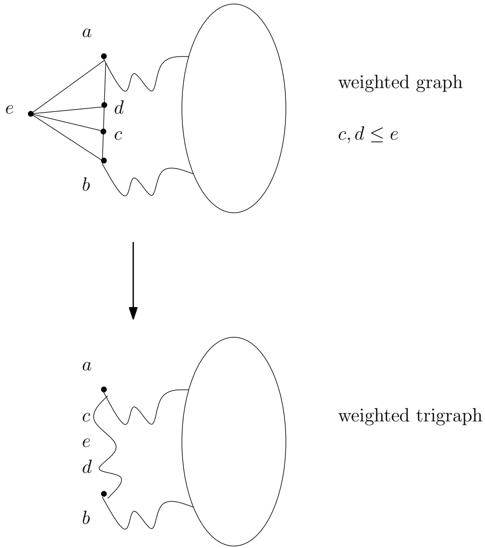
weight of S in (G, w):

$(1{+}3{+}11){+}5{+}17 = 37$

weight of S' in (G, w):

$(3 + 11) + (2 + 5) + 17 = 38$

Semi-adjacent pairs can imitate gems! (But without increasing the number of vertices, and without introducing wheels.)



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4. Then change weights in the other block, and (recursively) compute α.

That's all.

Thanks for listening!